CHAPTER 4: CIRCUIT THEOREMS

4.1 Linearity Property

- Linearity is the property of an element describing a linear relationship between cause and effect.
- Linearity property is a combination of the homogeneity (scaling) and the additivity properties.
- Scope of study – limits to resistors
- Homogeneity property:

If the input is multiplied by a constant, then the output is multiplied by the same constant.

From Ohm's law,

$$v = iR$$

If the current is increased by a constant k, then the voltage increases correspodingly by k.

$$kiR = kv$$

$\rightarrow$ refer to homogeneity property.

- Additivity property:

The response to a sum of inputs is the sum of the responses to each input applied separately.

Voltage-current relationship of a resistor

$$v_1 = i_1 R \qquad\qquad v_2 = i_2 R$$

then applying $(i_1 + i_2)$ gives

$$v = (i_1 + i_2)R = i_1 R + i_2 R = v_1 + v_2$$

$\rightarrow$ additivity property

- Thus, resistor is a linear element.
- Does the power relation is linear?

Consider the circuit as shown in Figure 4.1

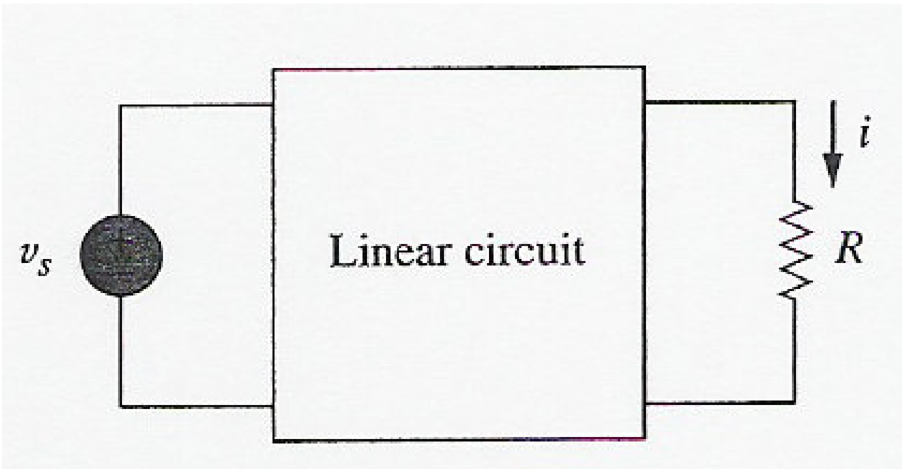

Figure 4.1

When current i_1 flows through resistor *R*,

$$p_1 = i_1^2 R$$

When current $i_2$flows through resistor *R*,

$$p_2 = i_2^2 R$$

If current $(i_1 + i_2)$ flows through resistor *R*,

$$p_3 = (i_1 + i_2)^2 R = R i_1^2 + R i_2^2 + 2R i_1 i_2 \neq p_1 + p_2$$

→ the relationship between power and voltage (or current) is nonlinear.

- Example:

For the circuit as shown in Figure 4.2, find I_0 when $v_s = 12$V and $v_s = 24$V.

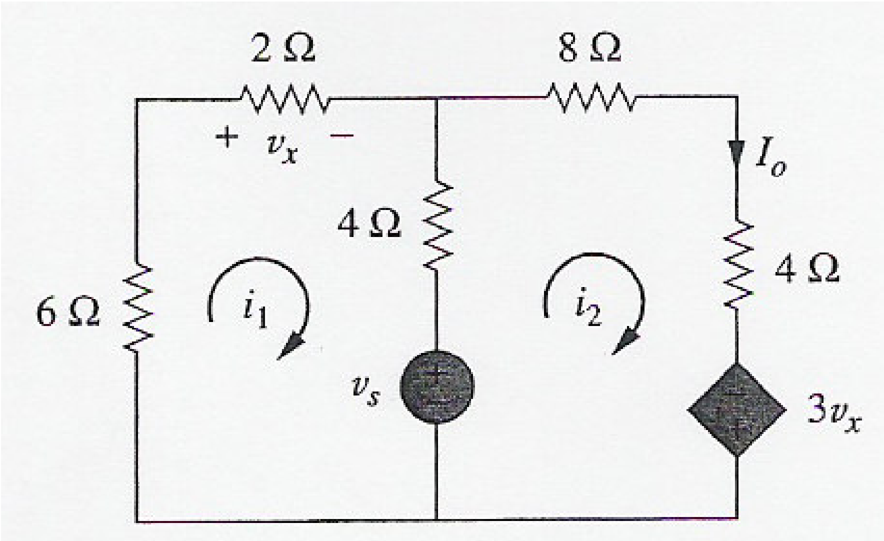

Figure 4.2

Applying KVL to the two loops,

$$12i_1 - 4i_2 + v_s = 0 \qquad \text{----(a)}$$

$$-4i_1 + 16i_2 - 3v_x - v_s = 0 \ \text{ but } \ v_x = 2i_1$$

$$\therefore -10i_1 + 16i_2 - v_s = 0 \qquad \text{----(b)}$$

(a) + (b),

$$2i_1 + 12i_2 = 0$$

$$i_1 = -6i_2 \qquad \text{----(c)}$$

(c) into (a),

$$-76i_2 + v_s = 0$$

$$i_2 = \frac{v_s}{76}$$

When $v_s = 12$ V,

$$I_0 = i_2 = \frac{12}{76}\text{A}$$

When $v_s = 24$ V,

$$I_0 = i_2 = \frac{24}{76}\text{A}$$

showing that when the source value is doubled, I_0 doubles.

- Exercise:

  Assume that $V_0 = 1$V and use linearity to calculate the actual value of V_0 in the circuit of Figure 4.3.

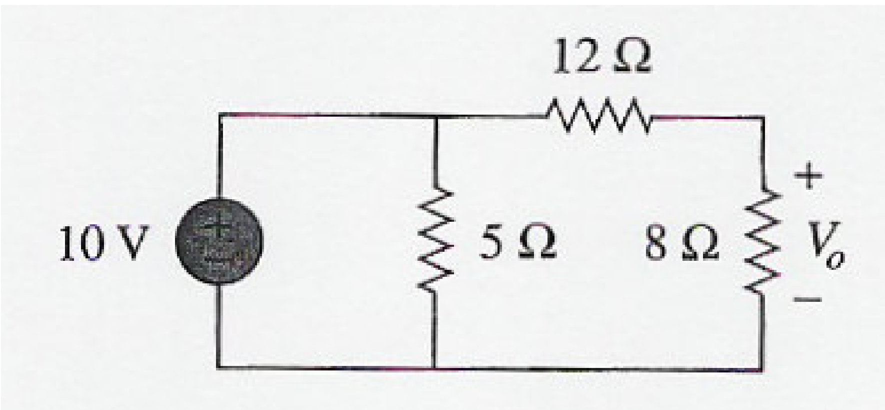

Figure 4.3

4.2 Superposition

- A way to determine the value of a specific variable (voltage or current).
- Superposition – determine the contribution of each independent source to the variable and then add them up.
- The idea of superposition rests on the linearity property.

> The **superposition** principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltages across (or currents through) that element due to each endependent source acting alone.

- Two main issues in superposition:
  (i) One independent source is considered at a time while all other independent sources are turned off. This implies that we replace every voltage source by 0 V (or a short circuit) and every current source by 0 A (or an open circuit).
  (ii) Dependent sources are left intact because they are controlled by circuit variables.
- Steps to aplly Superposition principle:

(i) Turn off all independent sources except one source. Find the output (voltage or current) due to that active using the previous techniques in Chapter 2 and 3.

(ii) Repeat step 1 for each of the other independent sources.

(iii) Find the total contribution by adding algebraically all the contributions due to the independent sources.

- Example 1:

  Use the superposition theorem to find *v* in the circuit as shown in Figure 4.4

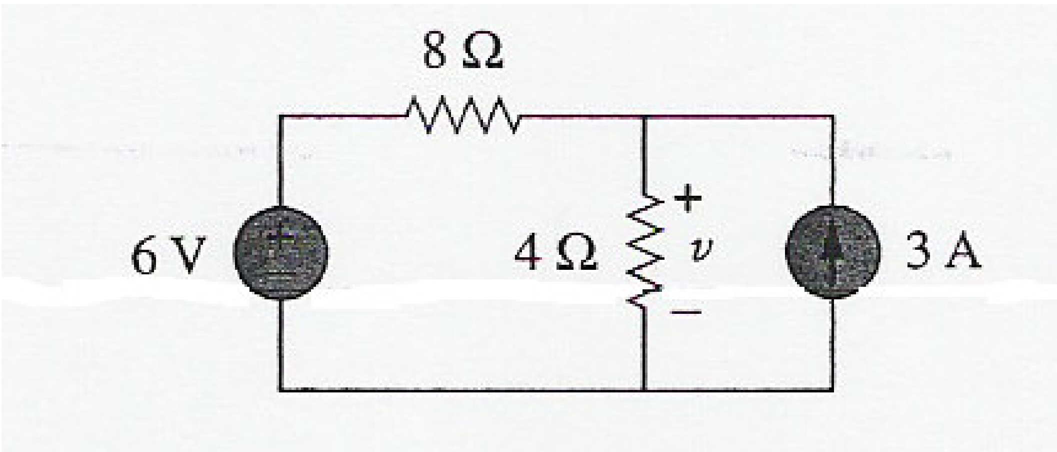

Figure 4.4

Since there are two sources, let

$$v = v_1 + v_2$$

where v_1 and v_2 are the contribution due to 6 V voltage source and the 3 A current source respectively.

To obtain v_1, set the current source to 0 A as shown in Figure 4.5.

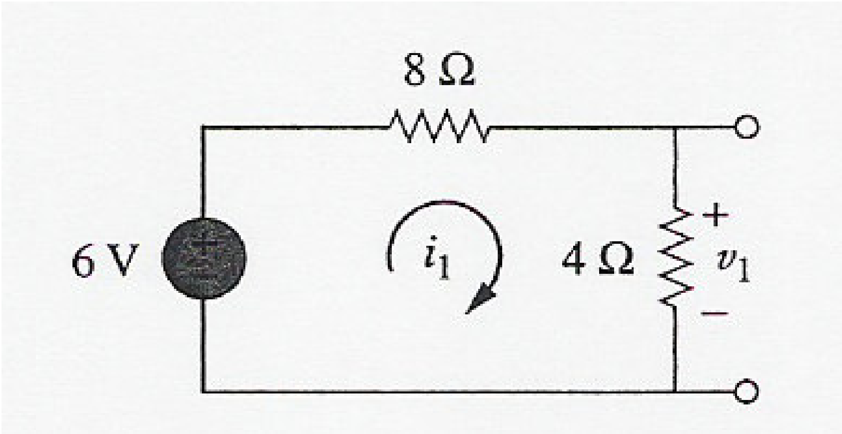

Figure 4.5

Applying KVL,

$$12i_1 - 6 = 0$$

$$i_1 = 0.5A$$

$$\therefore v_1 = 4i_1 = 2V$$

To obtain v_2, set the voltage source to 0 V as shown in Figure 4.6

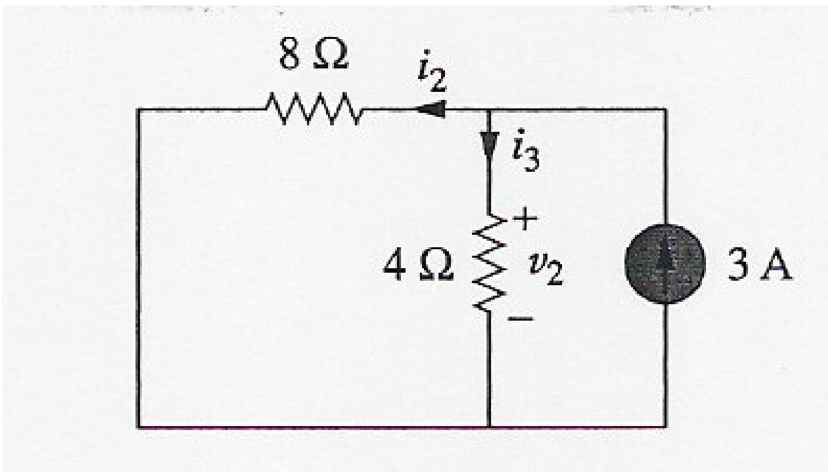

Figure 4.6

Using current division,

$$i_3 = \frac{8}{4+8}(3) = 2 \text{ A}$$

$$\therefore v_2 = 4i_3 = 8 \text{ V}$$

Thus,

$$v = v_1 + v_2 = 10 \text{ V}$$

- Example 2:

Find i_0 in the circuit in Figure 4.7 using superposition

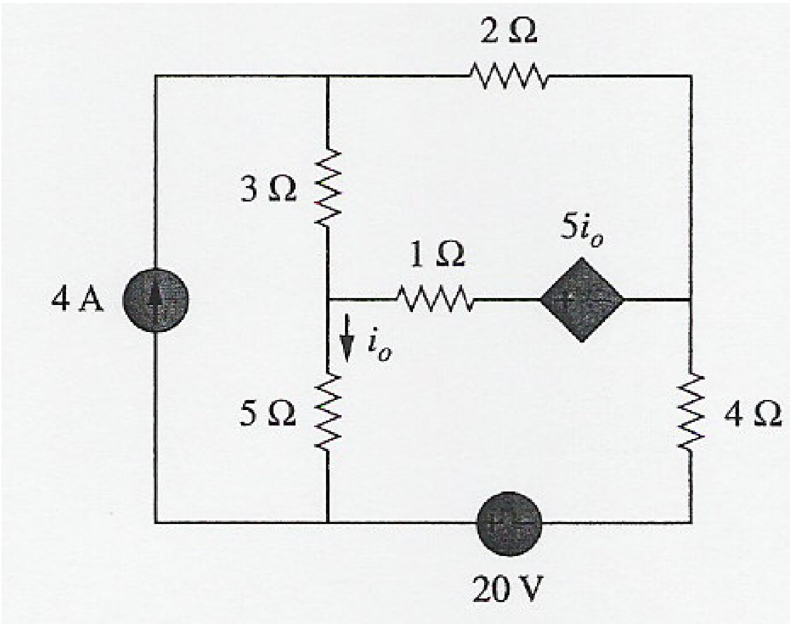

Figure 4.7

The circuit in Figure 4.7 involves a dependent source, which must be left intact.

We let,

$$i_0 = i_0' + i_0''$$

where i_0' and i_0'' are due to the 4 A current source and 20 V volatge source respectively.

To obtain i_0', we turn off the 20 V source as shown in Figure 4.8

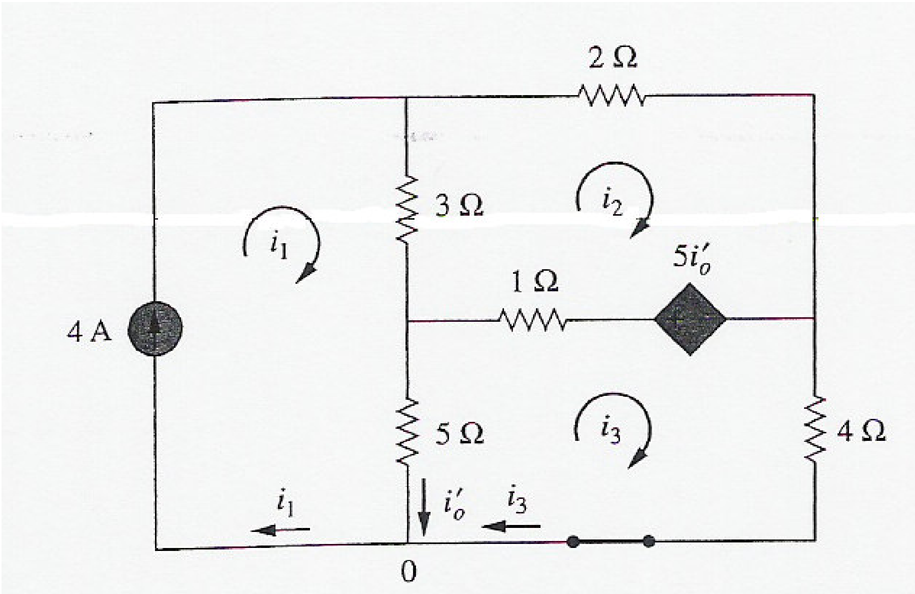

Figure 4.8

Apply mesh analysis,

For loop 1:

$$i_1 = 4\text{A}$$

For loop 2:

$$-3i_1 + 6i_2 - i_3 - 5i_0' = 0$$

For loop 3,

$$-5i_1 - i_2 + 10i_3 + 5i_0' = 0$$

At node 0,

$$i_3 = i_1 - i_0' = 4 - i_0'$$

From those equations, we get

$$3i_2 - 2i_0' = 8$$

$$i_2 + 5i_0' = 20$$

Thus,

$$i_0' = \frac{52}{17} \text{ A}$$

To obtain i_0'', we turn off the 4 A current source as shown in Fogure 4.9

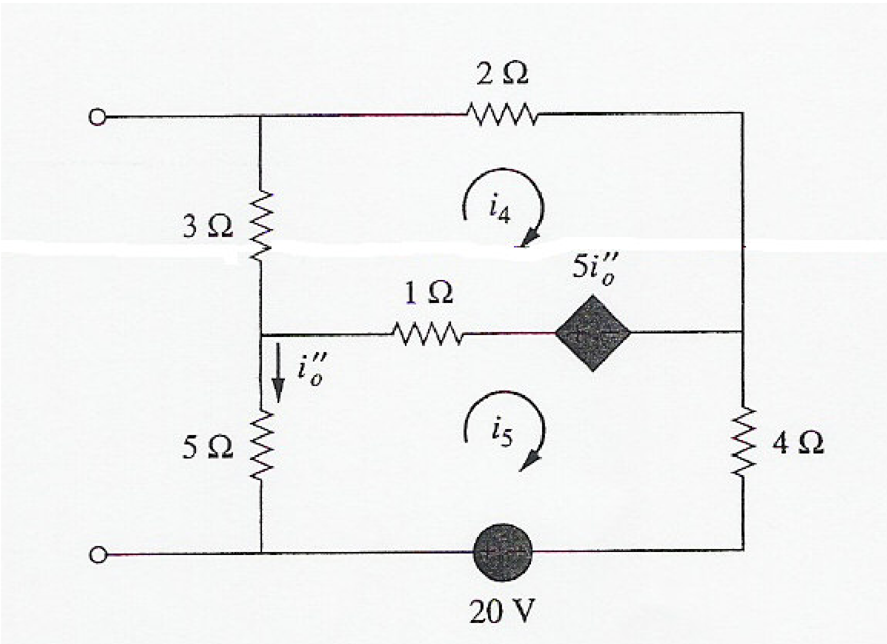

Figure 4.9

For loop 4, using KVL

$$6i_4 - i_5 - 5i_0'' = 0$$

For loop 5:

$$-i_4 + 10i_5 - 20 + 5i_0'' = 0$$

and $$i_5 = -i_0''$$

From these equations, we get

$$6i_4 - 4i_0'' = 0$$

$$i_4 + 5i_0'' = -20$$

$$\therefore i_0'' = -\frac{60}{17}\ \text{A}$$

Thus,

$$i_0 = -\frac{8}{17}\text{A}$$

4.3 Source Transformation

- A tool to simplify circuits.
- Definition:

> A **source transformation** is the process of replacing a voltage source v_s in series with a resistor R by a current source i_s in parallel with a resistor R, or vice versa.

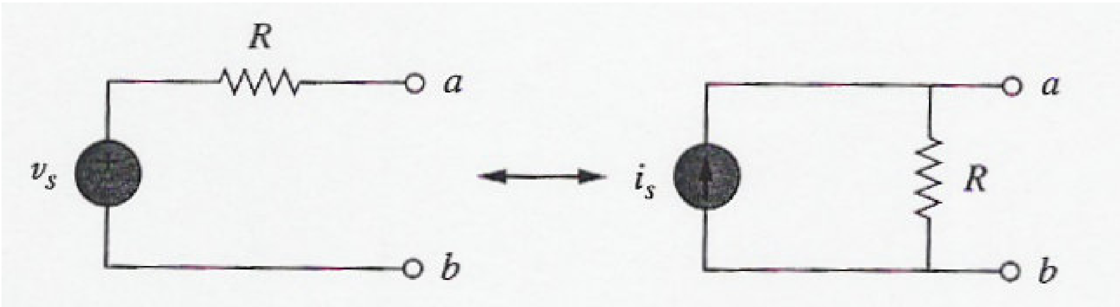

Figure 4.10

- Source transformation also applies to dependent sources.
- Two main issues in source transformation theorem:
  - (i) The arrow os the current source is directed toward the positive terminal of the volatge source (refer to Figure 4.10).
  - (ii) Source transformation is not possible when $R = 0$, which is the case with an ideal voltage source. Similarly, an ideal current source with

$R = \infty$ acnnot be replaced by a finite voltage source.

- Example 1:

Use source transformation to find v_0 in the circuit in Figure 4.11

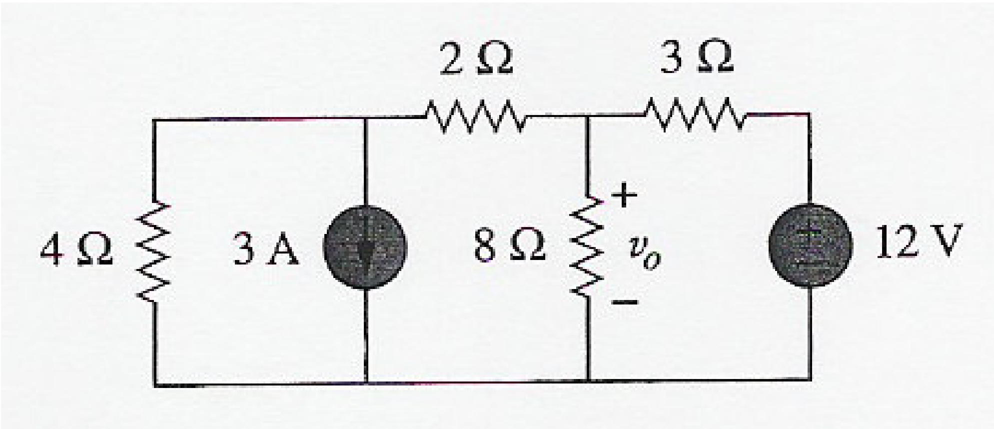

Figure 4.11

Transform the current and voltage sources to obtain the circuit,

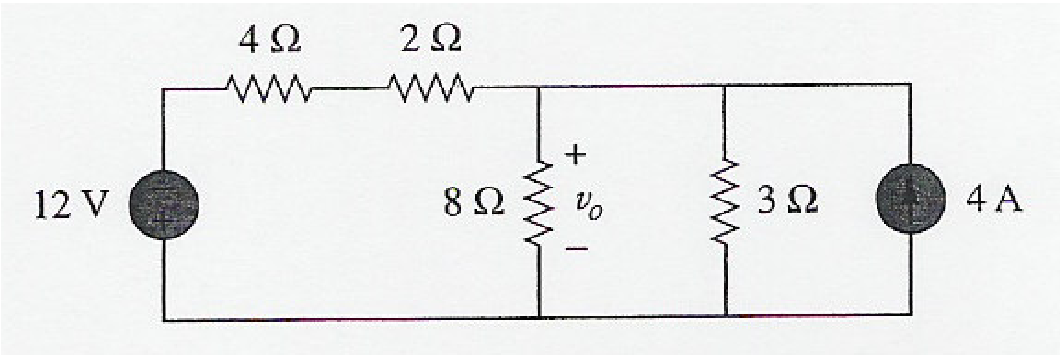

Figure 4.12

Combine 4 Ω and 2 Ω resistors → 6 Ω.

Transform the 12 V voltage source,

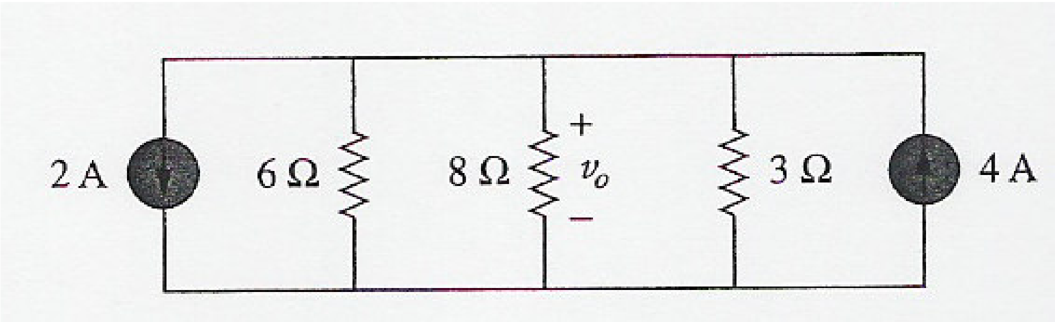

Figure 4.13

Combine 3 Ω and 6 Ω resistors in parallel → 2 Ω.

Combine 2 A and 4 A current source → 2 A,

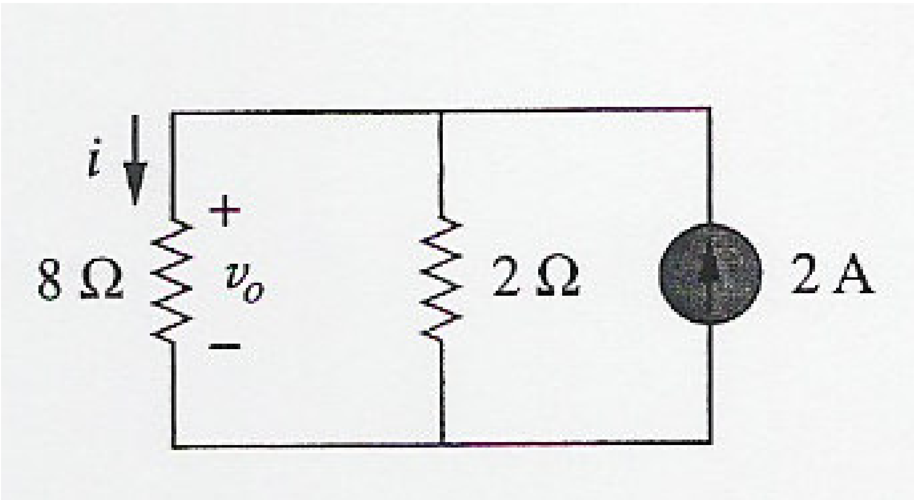

Figure 4.14

Use current division,

$$i = \frac{2}{2+8}(2) = 0.4 \text{ A}$$

and

$$v_0 = 8i = 8(0.4) = 3.2 \text{ V}$$

- Example 2:

Find v_x in Figure 4.15 using source transformation.

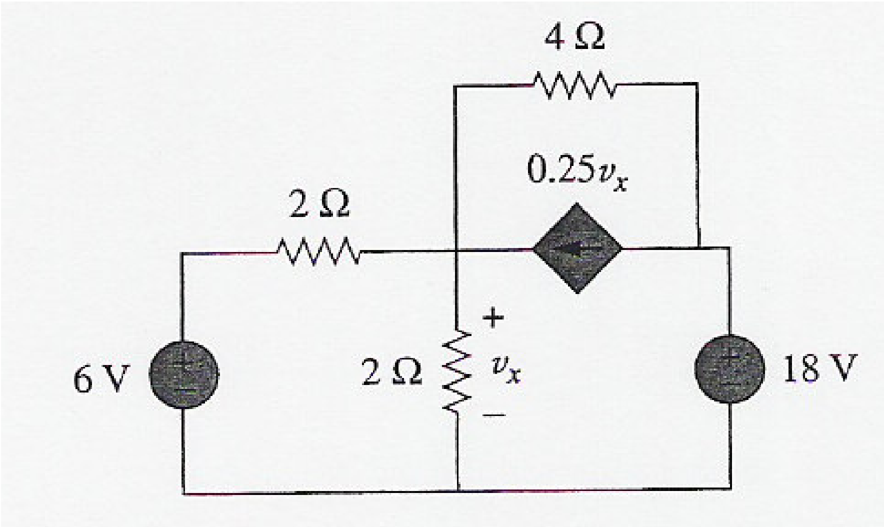

Figure 4.15

Transform the dependent current source and 6 V independent voltage source,

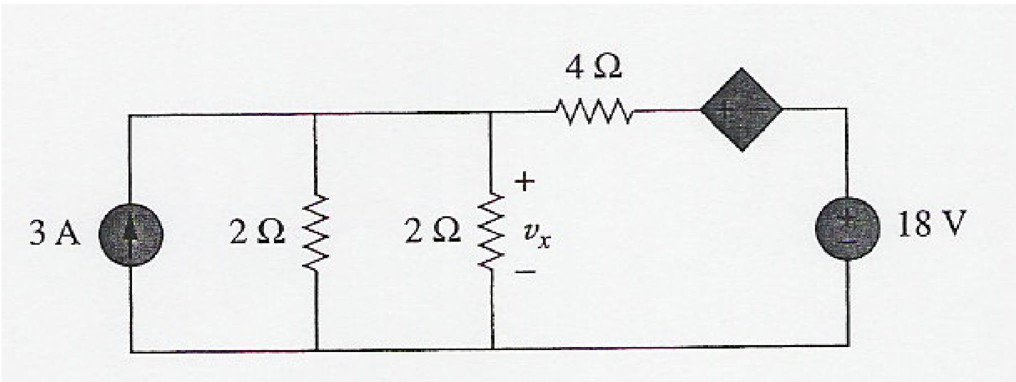

Figure 4.16

Combine the 2 Ω resistors in parallel → 1 Ω resistor which is parallel with 3 A current source.

Transform the 3 A current source,

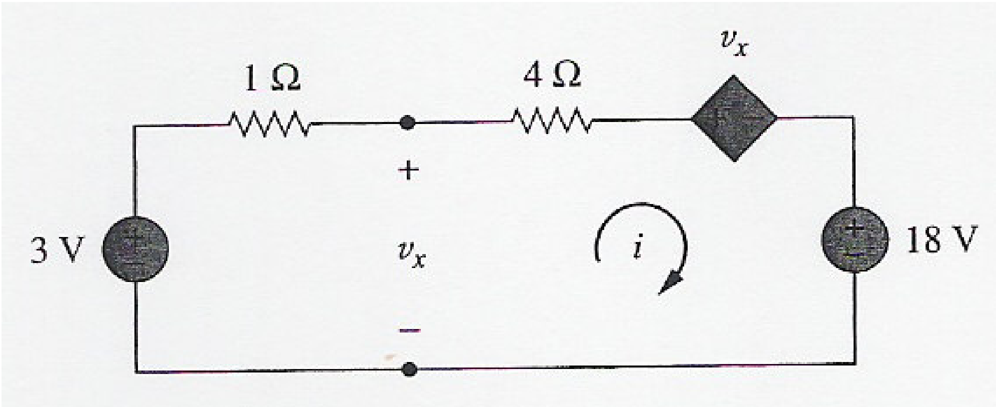

Figure 4.17

Applying KVL,

$$-3+5i+v_x+18=0$$

Applying KVL to the loop containing only the 3 V voltage source, 1 Ω resistor and v_x yields,

$$-3+i+v_x=0$$

$$\therefore v_x=3-i$$

Thus,

$$15+5i+3-i=0$$
$$i=-4.5 \text{ A}$$
$$\therefore v_x=3-(-4.5)=7.5 \text{ V}$$

4.4 Thevenin's Theorem

- Sometimes, a particular element is variable (called load) while other elements are fixed.

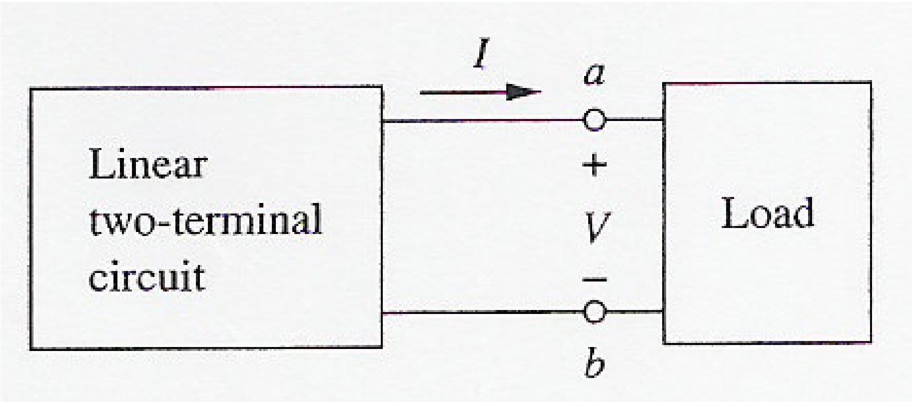

Figure 4.18

- If the variable element is changed, the circuit has to be analyzed all over again.
- Thevenin's theorem replaces the fixed part of the circuit with an equivalent circuit.

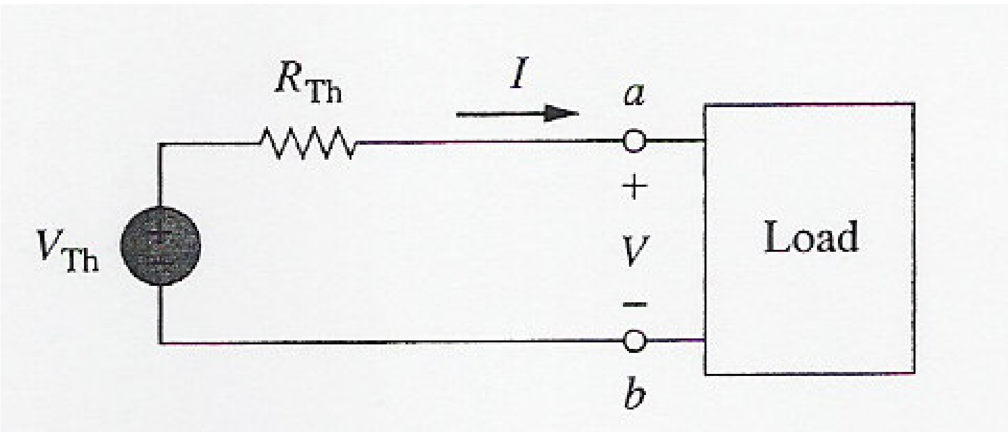

Figure 4.19

Note: The circuit to the left of the terminals *a-b* is known as the *Thevenin equivalent circuit*.

Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit of a voltage source V_{TH} in series with a resistor R_{TH}, where V_{TH} is the open-circuit voltage at the terminals and R_{TH} is the input or equivalent resistance at the terminals when the independent sources are turned off.

- Suppose the circuits in Figure 4.18 and 4.19 are equivalent (they have the same voltage-current ralation at their terminal).
- If the terminals *a-b* are made open-circuited, no current flows, so that the open-circuit voltage across the terminal *a-b* in Figure 4.18 must be equal to the voltage source V_{TH} in Figure 4.19, since the two circuits are equivalent.

  Thus,

$$V_{TH} = v_{oc}$$

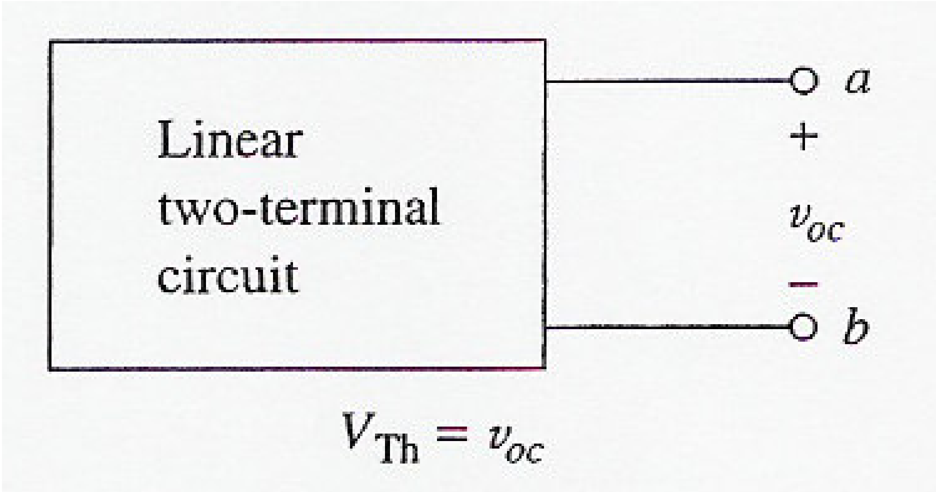

Figure 4.20

- Again, with the load disconnected and terminal *a-b* open-circuited, we turn off all independent sources.
- The input resistance (or equivalent resistance) of the dead circuit at the terminal a-b in Figure 4.18 must be equal to R_{TH} in Figure 4.19 because the two circuits are equivalent.

  Thus,

$$R_{TH} = R_{in}$$

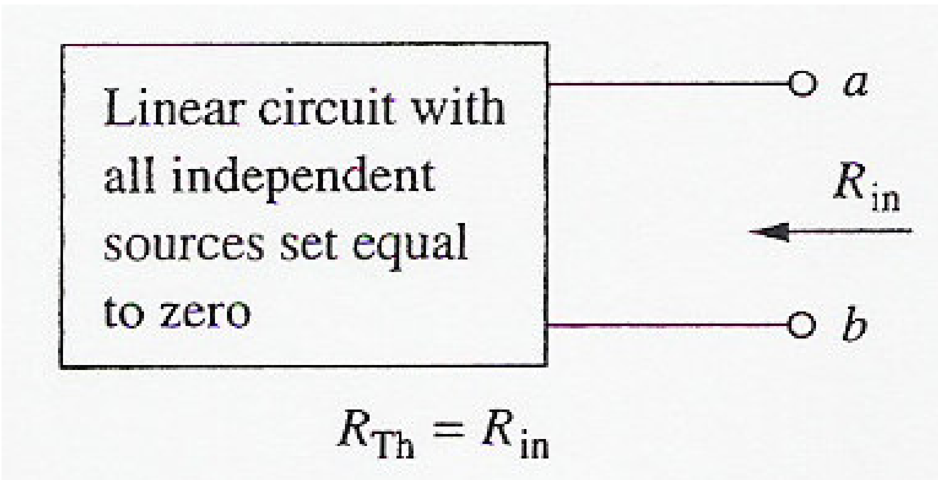

Figure 4.21

- Two cases must be considered (in finding for R_{TH}):

Case 1:

- If the network has no dependent sources, we turn off all independent sources.
- R_{TH} is the input resistance of the network looking between terminals *a* and *b* as shown in Figure 4.21.

Case 2:

- If the network has dependent sources, we turn off all independent sources.
- Apply a voltage source v_0 at terminals *a* and *b* and determine the resulting current i_0
- Then, $R_{TH} = \dfrac{v_0}{i_0}$

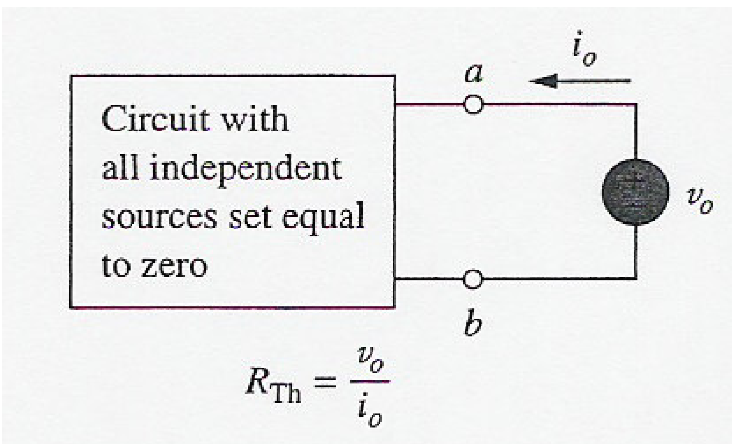

Figure 4.22

- Alternatively, we may insert a current source i_0 at the terminal a and b and find the terminal voltage v_0.
- Again, $R_{TH} = \frac{v_0}{i_0}$

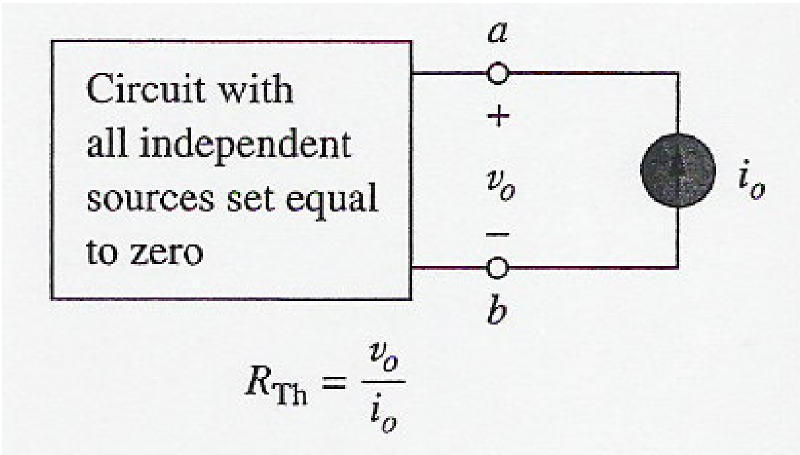

Figure 4.23

- Any value of v_0 and i_0 may be assumed.

• Consider the circuit as shown in Figure 4.24.

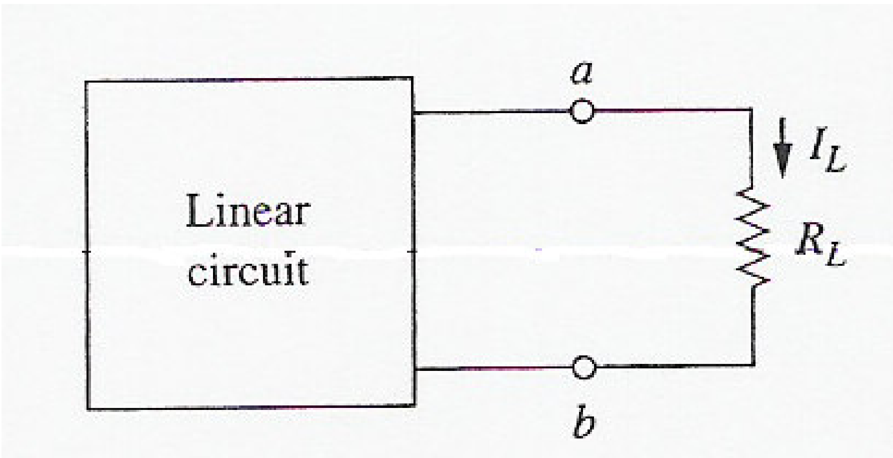

Figure 4.24

- After applying Thevenin's Theorem,

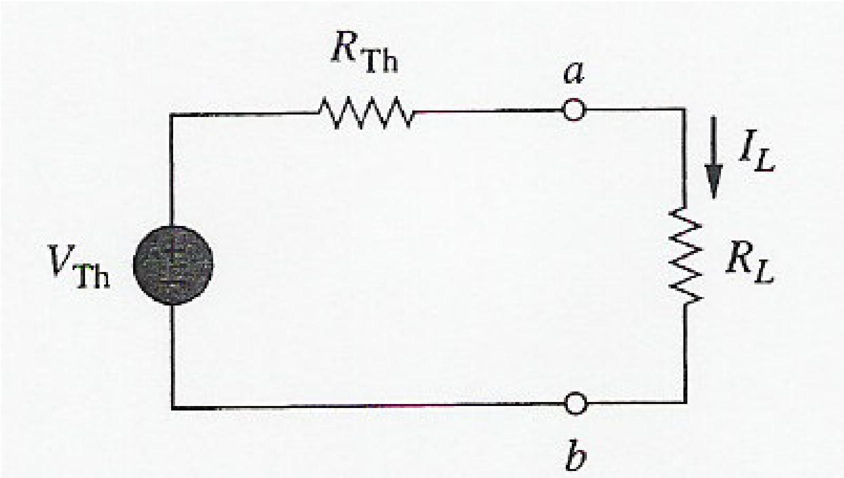

Figure 4.25

Thus,

$$I_L = \frac{V_{TH}}{R_{TH} + R_L}$$

$$V_L = R_L I_L = \frac{R_L}{R_{TH} + R_L} V_{TH}$$

- Example 1:
  Find the Thevenin equivalent circuit of the circuit as shown in Figure 4.26 to the left of the terminal *a-b*. Then find the current through $R_L = 6\Omega$.

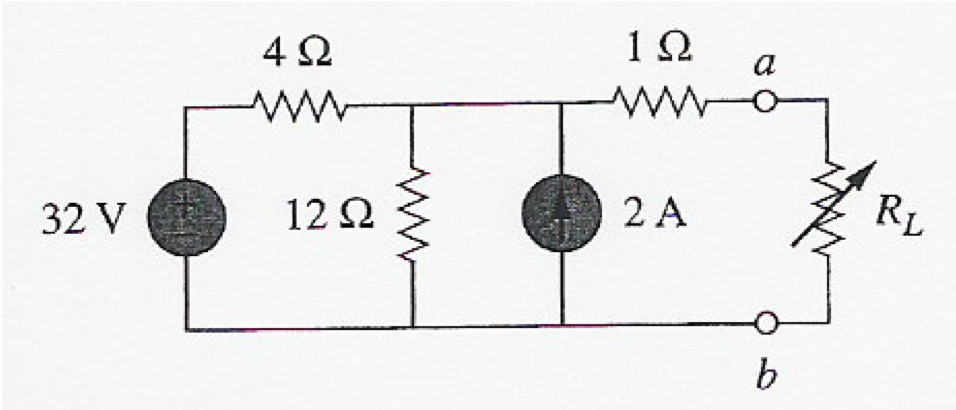

Figure 4.26

To find R_{TH}, turn off 32 V volatge source and 2 A current source,

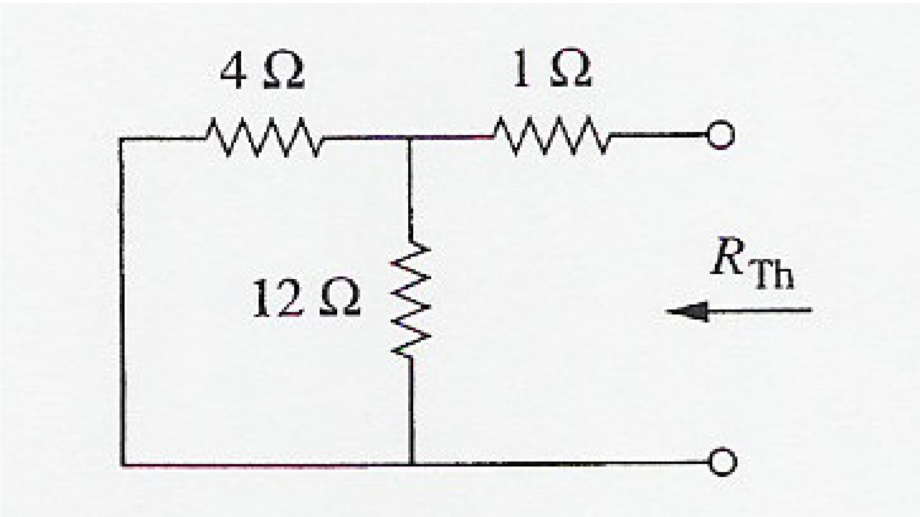

Figure 4.27

$$R_{TH} = 4 \| 12 + 1 = \frac{4 \times 12}{16} + 1 = 4\Omega$$

To find V_{TH},

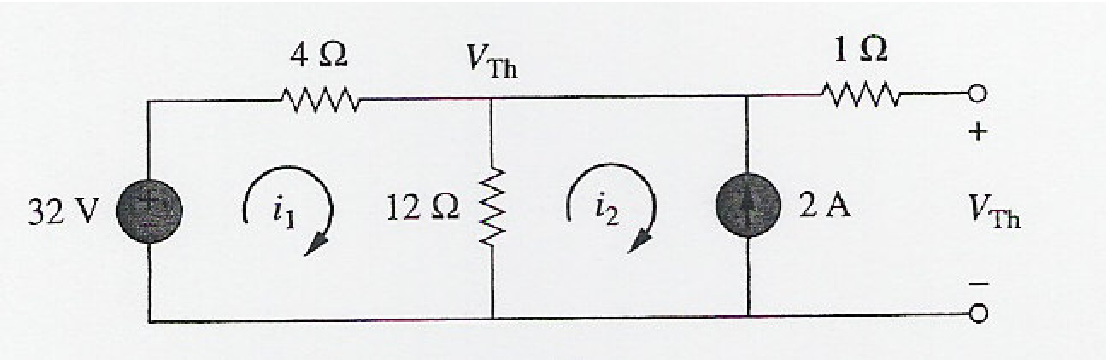

Figure 4.28

Applying mesh analysis,

Loop 1:

$$-32 + 4i_1 + 12(i_1 - i_2) = 0$$

Loop 2:

$$i_2 = -2 \text{ A}$$

Thus,

$$i_1 = 0.5 \text{ A}$$

$$\therefore V_{TH} = 12(i_1 - i_2) = 30 \text{ V}$$

When $R_L = 6\ \Omega$,

$$I_L = \frac{V_{TH}}{R_{TH} + R_L} = \frac{30}{10} = 3 \text{A}$$

- Example 2:

  Find the Thevenin equivalent circuit of the circuit as shown in Figure 4.29

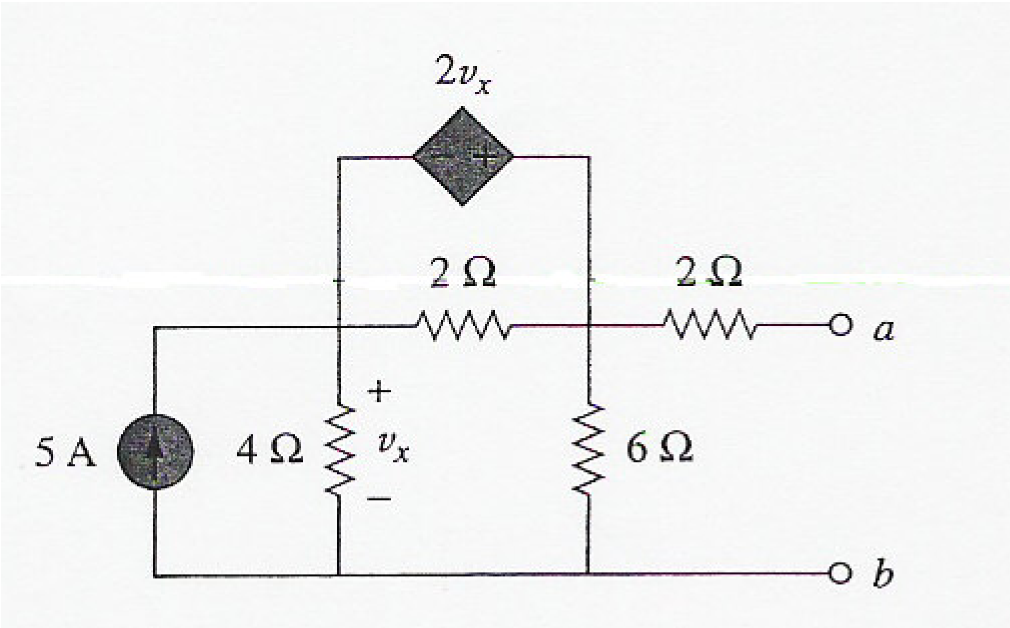

Figure 4.29

To find R_{TH}, set the independent source equal to zero and leave the dependent source alone.

Excite the network with a voltage source v_0 connected to the terminal and let $v_0 = 1$V.

Find i_0 and then obtain $R_{TH} = 1/i_0$.

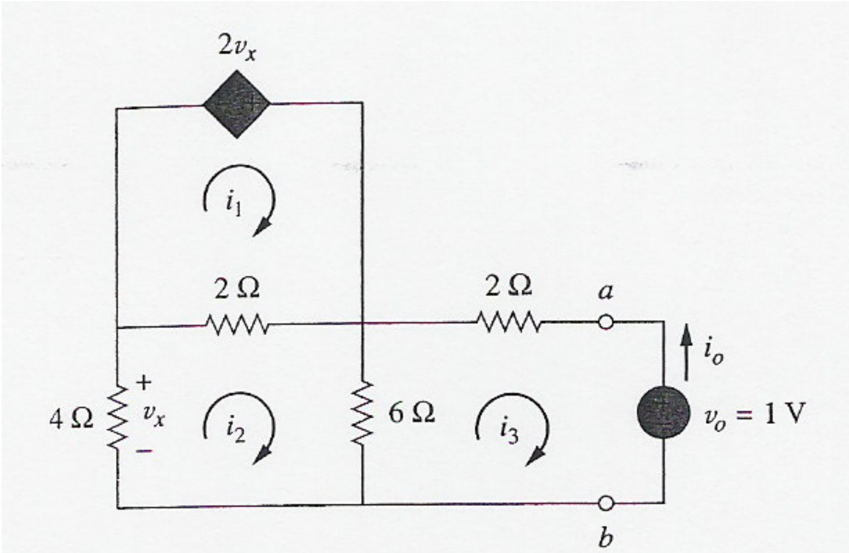

Figure 4.30

Loop 1: KVL

$$-2v_x + 2(i_1 - i_2) = 0$$

$$v_x = i_1 - i_2$$

From Loop 2:

$$-4i_2 = v_x = i_1 - i_2$$

$$\therefore i_1 = -3i_2$$

Loop 2:

$$4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0$$

Loop 3:

$$6(i_3 - i_2) + 2i_3 + 1 = 0$$

Solving these equations gives

$$i_3 = -\frac{1}{6}\ \text{A}$$

But

$$i_0 = -i_3 = 1/6 \text{ A.}$$

Hence,

$$R_{TH} = \frac{v_0}{i_0} = 6\Omega$$

To get V_{TH},

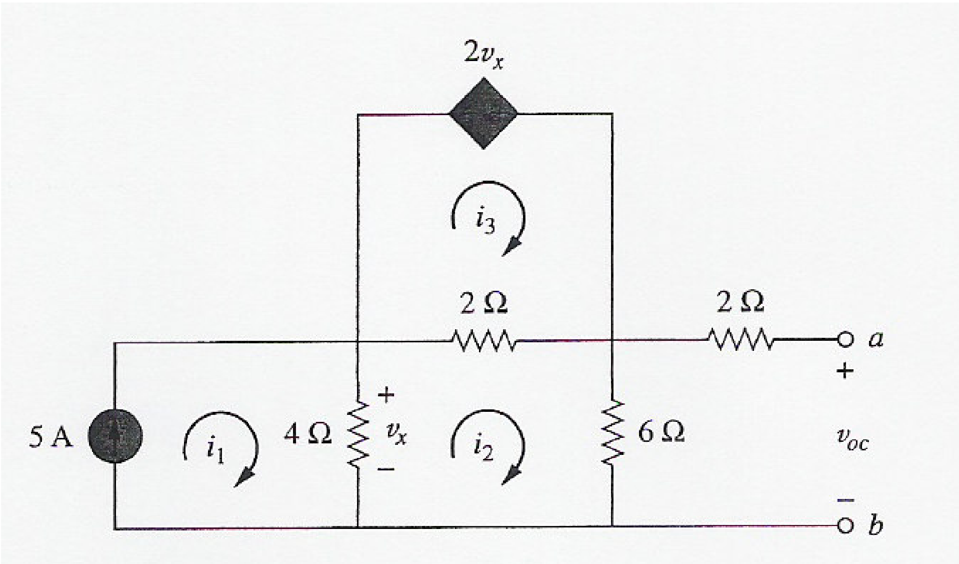

Figure 4.31

Applying mesh analysis:

$$i_1 = 5 \text{ A}$$

$$-2v_x + 2(i_3 - i_2) = 0$$

$$\therefore v_x = i_3 - i_2$$

$$4(i_2 - i_1) + 2(i_2 - i_3) + 6i_2 = 0$$

$$12i_2 - 4i_1 - 2i_3 = 0$$

But,

$$4(i_1 - i_2) = v_x$$

Solving those equations gives:

$$i_2 = 10/3 \text{ A}$$

Thus,

$$V_{TH} = v_{oc} = 6i_2 = 20 \text{ V}$$

4.5 Norton Theorem

- Definition:

> **Norton's Theorem** states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a current source I_N in parallel with a resistor R_N, where I_N is the short-circuit current through the terminals and R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

- Thus, the circuit in Figure 4.32 can be replaced by circuit in Figure 4.33.

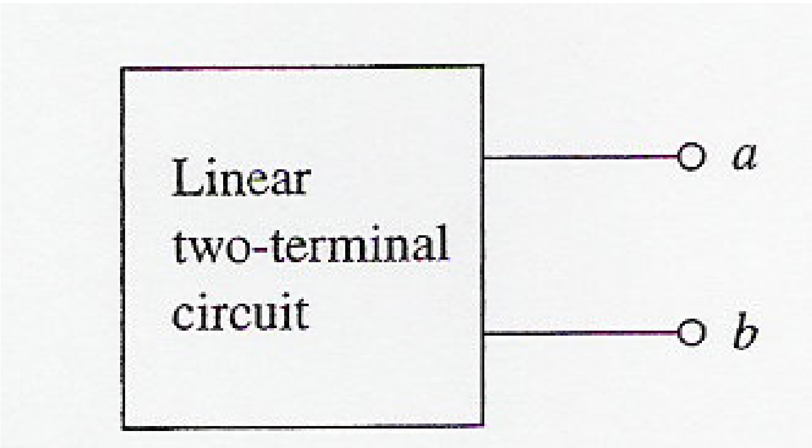

Figure 4.32

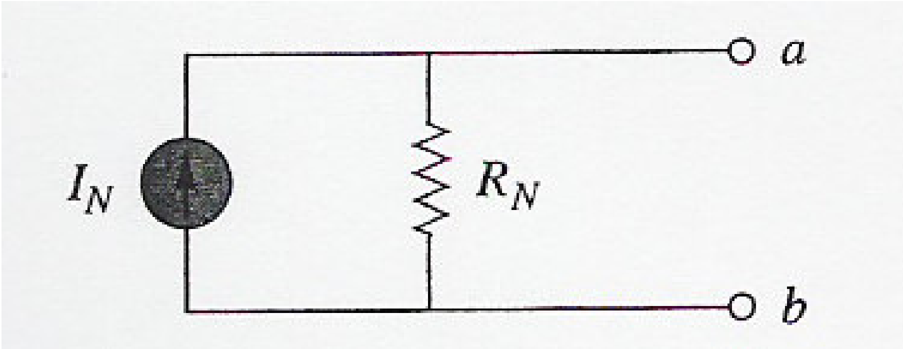

Figure 4.33

- The procedure to find R_N is the same way we find R_{TH}.

$$R_N = R_{TH}$$

- To find the Norton current, I_N we determine the short-circuit current flowing from terminal a to b (refer to Figure 4.32 and 4.33)

$$I_N = i_{sc}$$

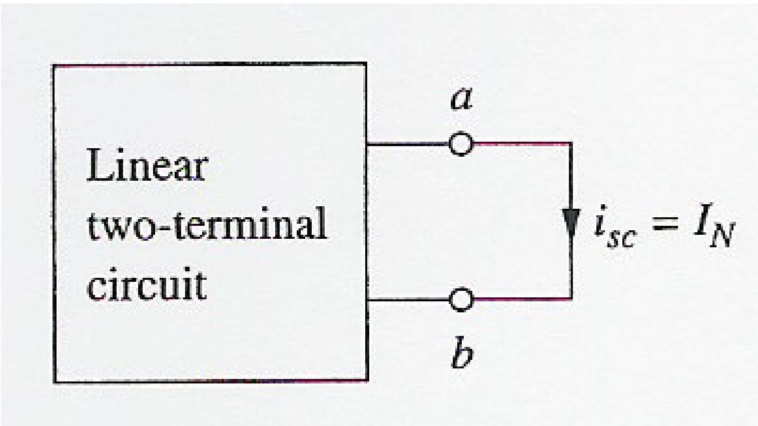

Figure 4.34

- Dependent and independent sources are treated the same way as in Thevenin's theorem.
- The relationship between Thevenin's and Norton's theorems:

$$I_N = \frac{V_{TH}}{R_{TH}}$$

- From the relationship, to determine the Thevenin and Norton equivalent circuit requires:
  - (i) The open-circuit voltage v_{oc} across terminals *a* and *b*.
  - (ii) The short-circuit current i_{sc} at terminals *a* and *b*.
  - (iii) The equivalent or input resistance R_{in} at the terminals *a* and *b* when all independent sources are turned off.
- As conclusion:

$$V_{TH} = v_{oc}$$

$$I_N = i_{sc}$$

$$R_{TH} = \frac{v_{oc}}{i_{sc}} = R_N$$

- Example:

  Find the Norton equivalent circuit of the circuit in Figure 4.35

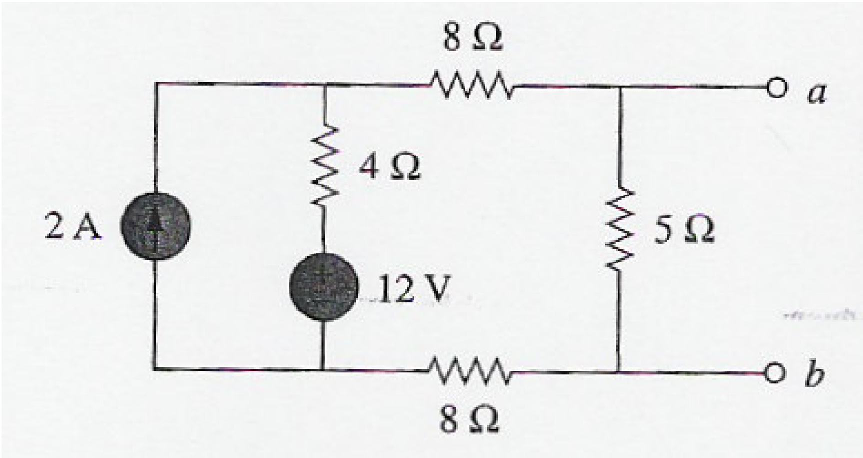

Figure 4.35

To find R_N, set the independent sources equal to zero,

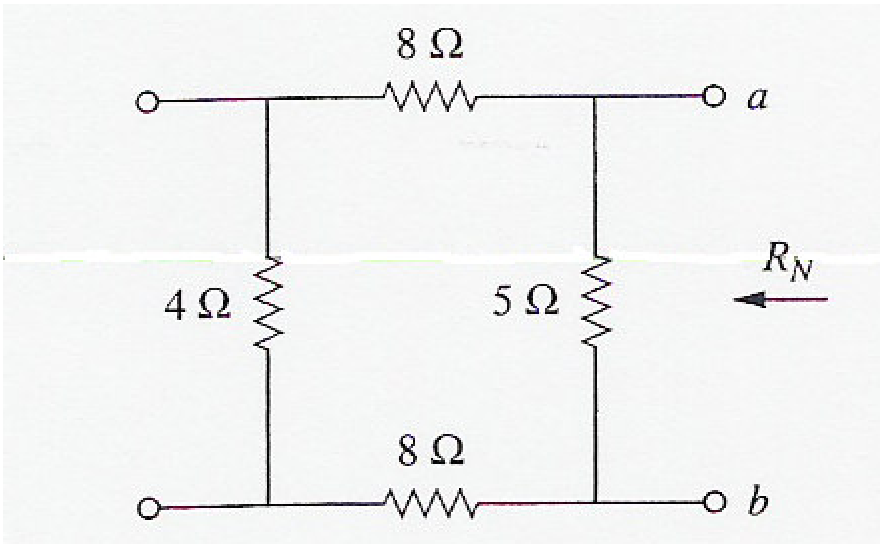

Figure 4.36

$$R_N = 5 \| (8+4+8) = 4\Omega$$

To find I_N, we short-circuit terminals *a* and *b*,

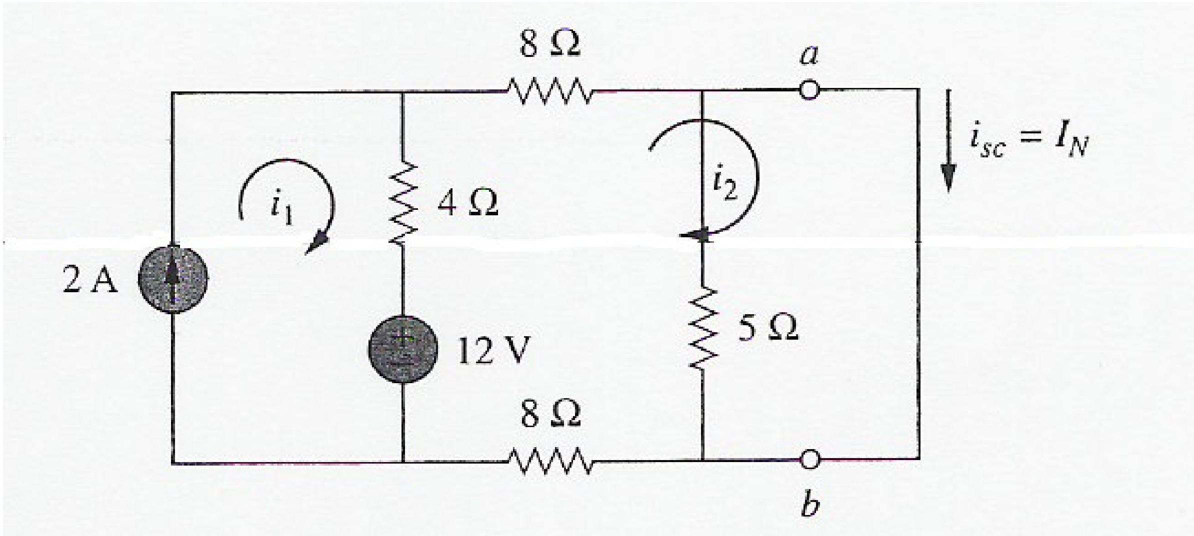

Figure 4.37

$$i_1 = 2\text{A}$$
$$20i_2 - 4i_1 - 12 = 0$$
$$\therefore i_2 = 1A = i_{sc} = I_N$$

- Example 2:
  Using Norton's theorem, find R_N and I_N of the circuit in Figure 4.38 at terminals *a* and *b*.

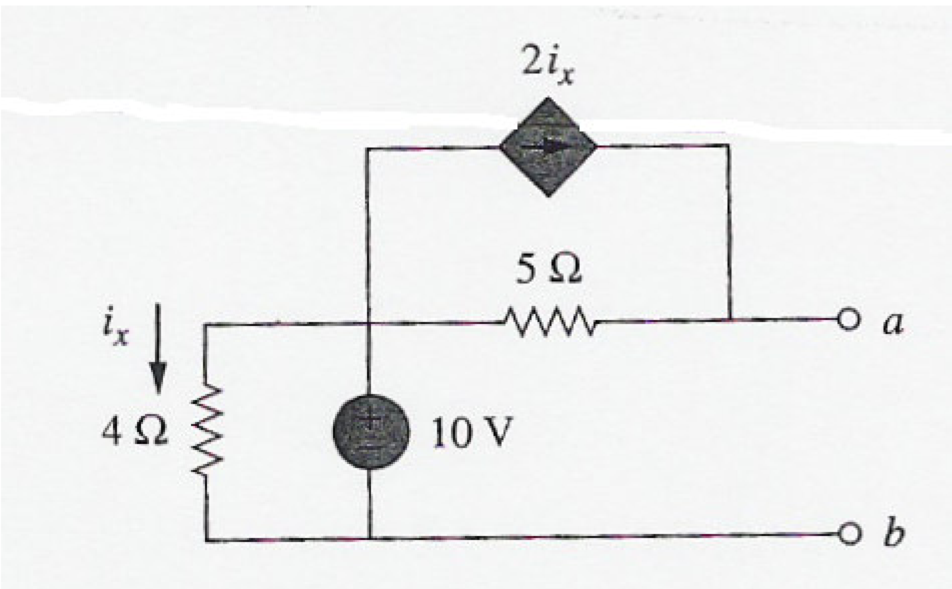

Figure 4.38

To find R_N, set the independent voltage source equal to zero and connect a voltage source of $v_0 = 1\text{V}$ (or any inspecified voltage v_0).

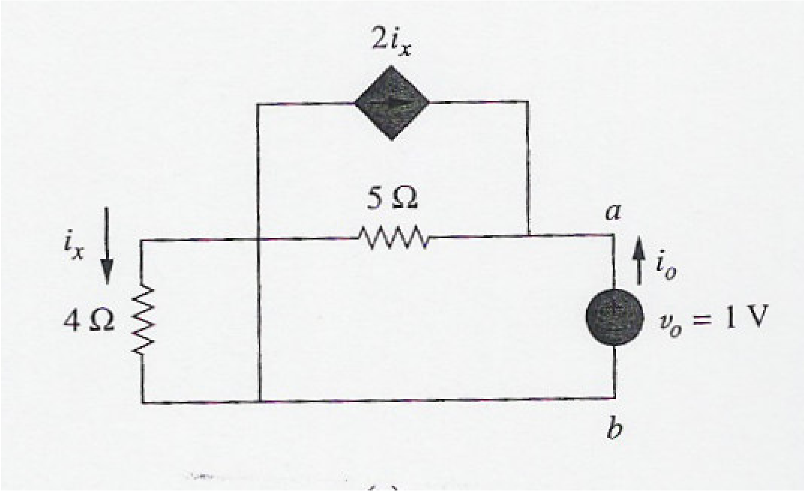

Figure 4.39

$$i_x = 0 \text{ A}$$

At node *a*,

$$i_0 = 5/1 = 5\text{A}$$

$$R_N = \frac{v_0}{i_0} = \frac{1}{5} = 0.2\Omega$$

To find I_N, we short-circuit terminals *a* and *b*, and find the current i_{sc},

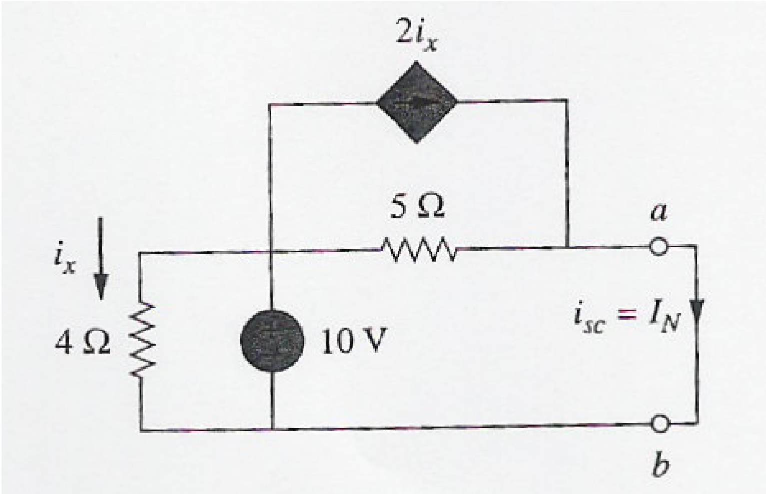

Figure 4.38

Note: All elements are in parallel.

$$i_x = \frac{10}{4} = 2.5 \text{ A}$$

At node *a*, KCL gives

$$i_{sc} = I_N = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\,\text{A}$$

4.6 Maximum Power Transfer

- The Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load.
- Consider a Thevenin equivalent circuit as shown in Figure 4.41

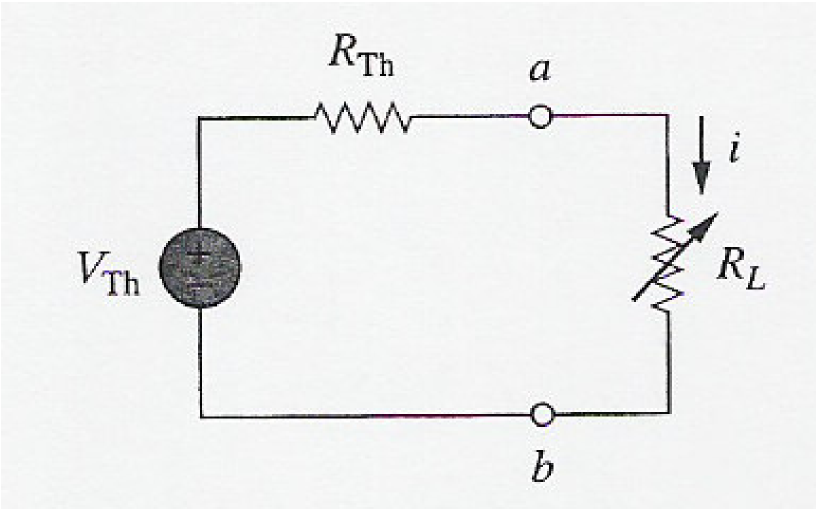

Figure 4.41

The power delivered to the load is,

$$p = i^2 R_L = \left(\frac{V_{TH}}{R_{TH} + R_L} \right)^2 R_L$$

- The V_{TH} and R_{TH} are fixed.
- By varying the load resistance R_L, the power delivered to the load varies

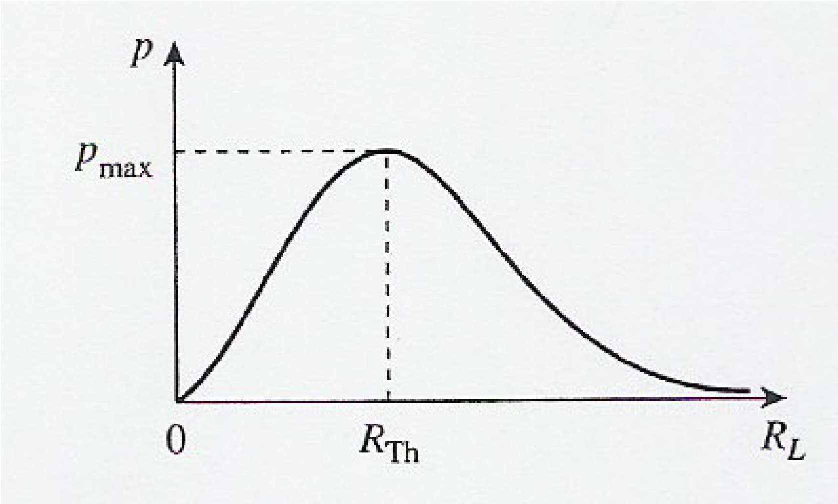

Figure 4.42

- From Figure 4.42, to find the maximum power, we differentiate p with respect to R_L and set the result equal to zero,

$$\frac{dp}{dR_L} = V_{TH}^2\left[\frac{(R_{TH}+R_L)^2 - 2R_L(R_{TH}+R_L)}{(R_{TH}+R_L)^4}\right] = 0$$

$$= V_{TH}^2\left[\frac{(R_{TH}+R_L-2R_L)}{(R_{TH}+R_L)^3}\right] = 0$$

This implies that,

$$0 = (R_{TH}+R_L-2R_L) = R_{TH} - R_L$$

Thus,

$$R_L = R_{TH} \text{ for maximum power delivered}$$

$$\therefore p_{\max} = \frac{V_{TH}^2}{4R_{TH}}$$